

# NONCOMMUTATIVE HYPERGEOMETRY

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ABSTRACT. A certain special function of the generalized hypergeometric variety is shown to fulfill a host of useful noncommutative identities.

## Introduction

Fix complex  $\tau$  with  $\text{Im } \tau > 0$  – so that  $q = e^{i\pi\tau}$  and  $q^{-1/\tau^2} = e^{-i\pi/\tau}$  both be less than 1 in modulus – and consider the function

$$(1) \quad \gamma(z) = \frac{(q^2 e^{-2i\pi z}; q^2)_\infty}{(e^{-2i\pi z/\tau}; q^{-2/\tau^2})_\infty},$$

where  $(a; b)_\infty$  is the usual Pochhammer-style symbol for  $(1-a)(1-ab)(1-ab^2)\dots$ , or explicitly

$$\gamma(z) = \frac{(1 - e^{-2i\pi(z-\tau)})(1 - e^{-2i\pi(z-2\tau)})(1 - e^{-2i\pi(z-3\tau)})\dots}{(1 - e^{-2i\pi z/\tau})(1 - e^{-2i\pi(z+1)/\tau})(1 - e^{-2i\pi(z+2)/\tau})\dots},$$

Clearly, this is meromorphic at all  $z \neq \infty$  and has a remarkable pattern of zeros and poles: all are simple, and, as shown on the figure, they fill the north-eastern

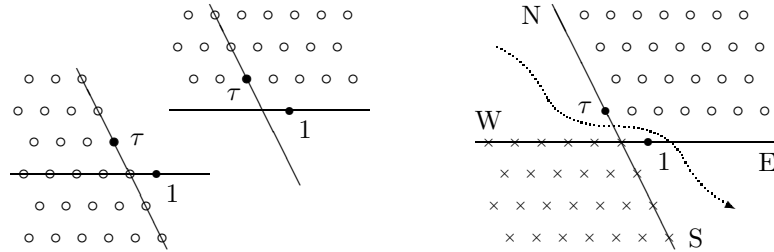


FIGURE 1. The denominator, numerator and whole of function  $\gamma$ . As an exercise, figure out what happens as  $\tau$  approaches the real line. Will it matter on which half it lands?

and south-western quarters –  $k, l > 0$  and  $k, l \leq 0$  respectively – of the lattice  $k + l\tau$ . With slight variations, this function, called below the  $\gamma$ -function, has of late been circulating in connection with quantum integrable models under fancy names like double sine or noncompact quantum dilogarithm.<sup>1</sup> Yet it deserves more attention than has so far been given, and so this article aims to introduce the matter to a wider mathphysical audience, and tie up a few loose ends in the process.

<sup>1</sup>See (Faddeev Kashaev Volkov 2001), (Kharchev Lebedev Semenov-Tian-Shansky 2003), (Bytsko Teschner 2003) and references therein. Ultimately, though, this function traces back to (Shintani 1977) and (Barnes 1899).

## 1. Two equations

Note that our function satisfies difference equations

$$(2) \quad \frac{\gamma(z+\tau)}{\gamma(z)} = 1 - e^{-2i\pi z} \quad \text{and} \quad \frac{\gamma(z+1)}{\gamma(z)} = 1 - e^{-2i\pi z/\tau},$$

and is indeed their only common solution – up to multiplication by an elliptic function of periods 1 and  $\tau$  that is, but this little ambiguity is easily avoided by adding a kind of minimal asymptotic condition that  $\gamma(z)$  goes to 1 as  $z$  goes southeast, that is  $z \rightarrow \infty$  in the sector  $\arg(-\tau) < \arg z < 0$ . So, these two equations can and will serve as a workable alternative to the infinite product definition. They also help explain the rather obscure title of this article.

On the one hand, each of the two resembles the most famous difference equation

$$\frac{\Gamma(z+1)}{\Gamma(z)} = z,$$

which defines Euler's Gamma function. Hence 'hypergeometry', and hence our function ought to be called, say, a double or elliptic gamma function – but since those names are already taken, let us settle for  $\gamma$ -function.

On the other hand, our two equations can be rewritten as

$$e^{-2i\pi z} \gamma(z) + \gamma(z+\tau) = \gamma(z) \quad \text{and} \quad e^{-2i\pi z/\tau} \gamma(z) + \gamma(z+1) = \gamma(z),$$

or in operator form

$$(2') \quad (e^{-2i\pi z} + e^{\tau d/dz})\gamma = \gamma \quad \text{and} \quad (e^{-2i\pi z/\tau} + e^{d/dz})\gamma = \gamma,$$

where, by usual abuse of notation,  $z$  and  $d/dz$  stand for operators of multiplication and differentiation by  $z$ , that is

$$(zf)(z) = zf(z) \quad \text{and} \quad \frac{d}{dz}f(z) = f'(z).$$

This identifies the  $\gamma$ -function as an eigenfunction of useful operators and so gives everything we do some sort of 'noncommutative' meaning. For instance, if those operators share one eigenfunction, might they also share the rest of them and be therefore functions of each other? Yes indeed, it is easily checked that functions

$$\psi_\lambda(z) = e^{-2i\pi\lambda z/\tau} \gamma(z-\lambda)$$

satisfy full spectral equations

$$(3) \quad (e^{-2i\pi z} + e^{\tau d/dz})\psi_\lambda = e^{-2i\pi\lambda} \psi_\lambda \quad \text{and} \quad (e^{-2i\pi z/\tau} + e^{d/dz})\psi_\lambda = e^{-2i\pi\lambda/\tau} \psi_\lambda.$$

Hence, by comparing the respective eigenvalues and optimistically assuming that functions  $\psi_\lambda$  span some reasonable functional space like  $L^2$  on the dotted line in Figure 1, follows a somewhat surprising relation

$$(4) \quad (e^{-2i\pi z} + e^{\tau d/dz})^{1/\tau} = e^{-2i\pi z/\tau} + e^{d/dz},$$

which has previously only been noticed in one particular case that  $1/\tau$  is positive integer. Let us not get ahead of ourselves, though, and get over with the hypergeometric part first.

## 2. Reflection formula

Recall the classical formula

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z},$$

which says that reflection about the point  $1/2$  reduces the Gamma function to an elementary one. The same happens to be true of the  $\gamma$ -function except the natural

reflection point is now  $(1 + \tau)/2$ . Indeed, since poles and zeros of the  $\gamma$ -function are symmetric to each other about the said point, the product

$$G(z) = \gamma(z)\gamma(1 + \tau - z)$$

has none of either. It might then equal a Gaussian exponential, and so it turns out. By one of equations (2) we have

$$\frac{G(z + \tau)}{G(z)} = \frac{\gamma(z + \tau)\gamma(1 - z)}{\gamma(z)\gamma(1 + \tau - z)} = \frac{1 - e^{-2i\pi z}}{1 - e^{-2i\pi(\tau - z)}} = -e^{-2i\pi z},$$

and by the other  $G(z + 1)/G(z) = \dots = -e^{2i\pi z/\tau}$ . Since the same equations are solved by  $e^{i\pi z(1 + \tau - z)/\tau}$ , we have

$$G(z) = e^{i\pi z(1 + \tau - z)/\tau} \times \text{an elliptic function with periods 1 and } \tau,$$

but without zeros nor poles on either side, that elliptic factor can only be constant. Set  $z = 1$  to show that that constant equals  $-\gamma(1)\gamma(\tau)$ , then set  $z = 0$  in equations (2) to show that

$$\gamma(\tau) = 2i\pi \operatorname{Res} \gamma(z)|_{z=0} = \tau \gamma(1).$$

Hence the anticipated ‘reflection formula’:

$$(5) \quad \gamma(z)\gamma(1 + \tau - z) = -\tau \gamma(1)^2 e^{i\pi z(1 + \tau - z)/\tau}.$$

This is good enough in this context, yet the question remains whether  $\gamma(1)$  could be evaluated in absolute terms. Those familiar with Dedekind’s eta function should already know because

$$\gamma(1) = \frac{(q^2; q^2)_\infty}{(q^{-2/\tau^2}; q^{-2/\tau^2})_\infty} = \frac{e^{-i\pi\tau/12} \eta(\tau)}{e^{i\pi/12\tau} \eta(-1/\tau)},$$

but the rest of us will have to wait until Section 4.

### 3. Under Fourier transform

Recall another classical formula called Euler’s Gamma integral, which reads

$$\Gamma(z) = \int_0^\infty \frac{dw}{w} w^z e^{-w},$$

and says that Fourier transform (or Mellin to be precise) reduces the Gamma function to the exponential one. The same cannot be quite true of the  $\gamma$ -function, for, as we remember, its defining equations (2’) mix differentiation and multiplication operators in a symmetric manner. So, since Fourier transform maps those operators more or less into each other, it will map the  $\gamma$ -function more or less into itself rather than reduce it to anything else. Specifically, consider the Fourier integral

$$\hat{\gamma}(z) = -\frac{1}{\tau} \int_{\text{NW}}^{\text{SE}} d\zeta e^{-2i\pi\zeta z/\tau} \gamma(\zeta)$$

along that same dotted line in Figure 1. Now one of equations (2’) translates into

$$\hat{\gamma}(z) = \frac{1}{\tau} \int_{\text{NW}}^{\text{SE}} d\zeta e^{-2i\pi\zeta z/\tau} (e^{-2i\pi\zeta/\tau} + e^{d/d\zeta}) \gamma(\zeta) = (e^{d/dz} + e^{2i\pi z/\tau}) \hat{\gamma}(z),$$

and the other into  $\hat{\gamma} = \dots = (e^{\tau d/dz} + e^{2i\pi z}) \hat{\gamma}(z)$ , which are, as expected, virtually the same equations as (2’) themselves. Clearly, their general solution is

$$\hat{\gamma}(z) = \frac{\text{an elliptic function with periods 1 and } \tau}{\gamma(1 + \tau - z)},$$

but since our integral converges too well for elliptic functions to creep in, that elliptic factor is again constant – and equals, of course,  $(2i\pi/\tau) \operatorname{Res} \gamma(z)|_{z=0}$ , that is, as

we already know, that same  $\gamma(1)$ . Hence, after inversion, follows the ‘tau-gamma integral’

$$(6) \quad \gamma(z) = \gamma(1) \int_{\text{NW}}^{\text{SE}} d\zeta \frac{e^{2i\pi\zeta z/\tau}}{\gamma(1+\tau-\zeta)},$$

which confirms that the  $\gamma$ -function is indeed a Fourier image of more or less itself rather than some quasiexponential function. Or is it that the  $\gamma$ -function somehow emulates both the gamma and exponential functions? We will find out in Section 6.

#### 4. Tau-binomial theorem

The more immediate question is, what about the beta integral? This is settled by Kashaev-Ponsot-Teschner’s ‘tau-binomial theorem’: for all  $y \neq k+l\tau$  ( $\mathbb{Z} \ni k, l \leq 0$ ) we have

$$(7) \quad \frac{\gamma(y)\gamma(z)}{\gamma(y+z)} = \gamma(1) \int_{\text{NW}}^{\text{SE}} d\zeta \frac{e^{2i\pi\zeta z/\tau} \gamma(y-\zeta)}{\gamma(1+\tau-\zeta)},$$

provided the middle part of the integration line is rerouted, if necessary, so as to separate zeros of  $\gamma(1+\tau-\zeta)$  from poles of  $\gamma(y-\zeta)$ .<sup>2</sup> This is a straightforward extension of the tau-gamma integral (6), in the sense that it reduces to the latter as  $y$  goes southeast, and is verified along the same lines, that is by comparing equations with respect to  $z$  and evaluating contribution of the pole at  $\zeta = 0$ . The next question, then, is whether further extension is possible. The short answer is no, there are no more Fourier integrals left to take. A longer answer follows in the next section, but to wrap this one up, let us consider another two useful limit cases.

First, send  $y+z$  to zero to obtain in the limit

$$\delta(y) = -\frac{1}{\tau} \int_{\text{NW}}^{\text{SE}} d\zeta \frac{e^{-2i\pi\zeta y/\tau} \gamma(y-\zeta)}{\gamma(1+\tau-\zeta)},$$

or, after a suitable change of variables,

$$(8) \quad \delta(z-y) = \int_{\text{NW}}^{\text{SE}} d\lambda \psi^\lambda(z) \psi_\lambda(y),$$

where  $\psi_\lambda(y) = e^{-2i\pi\zeta y/\tau} \gamma(y-\zeta)$  are those same (generalized) eigenfunctions from Section 1, and  $\psi^\lambda(z) = (-1/\tau) e^{2i\pi\lambda z/\tau} / \gamma(1+\tau-\lambda+z)$ . I leave it to the reader to figure out the details, but the upshot is, anyway, that functions  $\psi_\lambda$  do indeed span a wide range of functional spaces, which consist of functions that are, loosely speaking, well-behaved in the northwestern and southeastern quarters (as mapped in Figure 1) of the complex plane. This bodes well for potential ‘noncommutative’ applications, but, again, let us get over with the hypergeometric part first.

Second, apply the reflection formula (5) a few times and change variables so that the tau-binomial theorem (7) becomes

$$(7') \quad \frac{\gamma(1+\tau-z)\gamma(y)}{\gamma(1+\tau-z+y)} = -\tau\gamma(1)^3 \int_{\text{NW}}^{\text{SE}} d\zeta \frac{e^{i\pi\zeta(1+\tau-\zeta)/\tau}}{\gamma(1+\tau-z+\zeta)\gamma(1+\tau-\zeta+y)},$$

then send both  $y$  and  $-z$  southeast to reduce things to the Gaussian integral

$$1 = -\tau\gamma(1)^3 \int_{\text{NW}}^{\text{SE}} d\zeta e^{i\pi\zeta(1+\tau-\zeta)/\tau}.$$

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<sup>2</sup>The same rule applies, without further mention, to all the integrals below: the integration line must always separate the southwestern and northeastern ‘sequences’ of poles.

Hence an opportunity to evaluate  $\gamma(1)$  without Dedekind's help. Take the integral and then, in order to pick the right cubic root, go back to the infinite product expansion (1) and see that  $\gamma(1)$  should equal 1 if  $\tau = i$ . Thus,

$$(9) \quad \gamma(1) = i^{5/6} \tau^{-1/2} e^{-i\pi(1+\tau)^2/12\tau},$$

but I am not sure what to make of it, and will be using  $\gamma(1)$  for shorthand anyway.

## 5. Beyond Fourier transform

Write

$$\frac{\gamma(z - \zeta)}{\gamma(1 + \tau - \zeta)} = \frac{\gamma(y + z - \zeta)}{\gamma(1 + \tau - \zeta)} \frac{\gamma(z - \zeta)}{\gamma(y + z - \zeta)}$$

and take Fourier transform of both sides (using the tau-binomial theorem (7) twice directly and once in reverse, and the fact that Fourier transform of a product is a convolution of Fourier transforms of its factors) to obtain

$$\frac{\gamma(x)\gamma(y)\gamma(z)}{\gamma(x+z)\gamma(y+z)} = \gamma(1) \int_{\text{NW}}^{\text{SE}} d\zeta \frac{e^{2i\pi\zeta z/\tau} \gamma(x - \zeta)\gamma(y - \zeta)}{\gamma(x + y + z - \zeta)\gamma(1 + \tau - \zeta)}.$$

This looks very like the  $\tau$ -binomial theorem (7) and obviously reduces to the latter as either  $x$  or  $y$  go southeast – yet the right hand side is no longer a Fourier integral of course. So, the longer answer to the question of the previous section is that extension of the tau-binomial theorem (7) is possible after all, but it turns out to be an ‘addition theorem’ rather than Fourier integral. How about another extension then?

Use the reflection formula (5) a few times and change variables so that the above formula takes a more transparent form

$$(10) \quad \frac{\gamma(\nu + \mu)\gamma(\nu + \kappa)\gamma(\mu)\gamma(\kappa)}{\gamma(\nu + \mu + \kappa)} \\ = -\frac{1}{\tau\gamma(1)} \int_{\text{NW}}^{\text{SE}} d\zeta e^{2i\pi\zeta(\nu+\zeta)/\tau} \gamma(\nu + \zeta)\gamma(\mu - \zeta)\gamma(\kappa - \zeta)\gamma(\zeta),$$

in which it unmistakably resembles the addition theorem for binomial coefficients<sup>3</sup>

$$\frac{(n + m + k)!}{(n + m)!(n + k)!m!k!} = \sum_j \frac{1}{(n + j)!(m - j)!(k - j)!j!}.$$

Then, since the latter is known to have exactly one extension in the shape of Pfaff-Saalschütz's sum

$$\frac{(n + l + k)!(n + m + k)!(n + m + l)!}{(n + m)!(n + l)!(n + k)!m!l!k!} = \sum_j \frac{(n + m + l + k - j)!}{(n + j)!(m - j)!(l - j)!(k - j)!j!}.$$

it is a safe educated guess that the  $\gamma$ -function satisfies a similar ‘ultimate integral identity’

$$(11) \quad \frac{\gamma(\nu + \mu)\gamma(\nu + \lambda)\gamma(\nu + \kappa)\gamma(\mu)\gamma(\lambda)\gamma(\kappa)}{\gamma(\nu + \lambda + \kappa)\gamma(\nu + \mu + \kappa)\gamma(\nu + \mu + \lambda)} \\ = -\frac{1}{\tau\gamma(1)} \int_{\text{NW}}^{\text{SE}} d\zeta e^{2i\pi\zeta(\nu+\zeta)/\tau} \frac{\gamma(\nu + \zeta)\gamma(\mu - \zeta)\gamma(\lambda - \zeta)\gamma(\kappa - \zeta)\gamma(\zeta)}{\gamma(\nu + \mu + \lambda + \kappa - \zeta)}$$

– at least safe enough not to bother verifying it just yet. We will do it anyway in the noncommutative part, which now begins.

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<sup>3</sup>Just in case, this expresses equality of coefficients of  $w^m$  in both hand sides of the formula  $(1 + w)^{n+m+k} = (1 + w)^{n+m}(1 + w)^k$ .

## 6. Going noncommutative

Let us go back to the tau-binomial theorem to try and interpret it as an operator relation. The form (7') is best suited for that. Apply the reflection formula (5) one more time to obtain

$$(7'') \quad \gamma(1 + \tau - z) \hat{\gamma}(z - y) \gamma(y) = \int_{\text{NW}}^{\text{SE}} d\zeta \hat{\gamma}(z - \zeta) \gamma(1 + \tau - \zeta) \gamma(\zeta) \hat{\gamma}(\zeta - y),$$

where, as before,  $\hat{\gamma}(z) = \gamma(1)/\gamma(1 + \tau - z)$  is Fourier image of the  $\gamma$ -function. This indeed lends itself to be interpreted as operator relation

$$abc = bacb,$$

where  $a$  and  $c$  are operators of pointwise multiplication by  $\gamma(1 + \tau - z)$  and  $\gamma(z)$ ,

$$af(z) = \gamma(1 + \tau - z)f(z) \quad \text{and} \quad cf(z) = \gamma(z)f(z),$$

and  $b$  is that of convolution with  $\hat{\gamma}$ ,

$$bf(z) = \int_{\text{NW}}^{\text{SE}} d\zeta \hat{\gamma}(z - \zeta) f(\zeta).$$

So if  $z$  and  $d/dz$  stand, as before, for operators of multiplication and differentiation by  $z$ , then, by near tautology,

$$a = \gamma(1 + \tau - z) \quad \text{and} \quad c = \gamma(z),$$

and, by a textbook argument about multiplication vs convolution,

$$\begin{aligned} \int_{\text{NW}}^{\text{SE}} d\zeta \hat{\gamma}(z - \zeta) f(\zeta) &= \int_{\text{NW}}^{\text{SE}} d\zeta \hat{\gamma}(\zeta) f(z - \zeta) \\ &= \int_{\text{NW}}^{\text{SE}} d\zeta \hat{\gamma}(\zeta) e^{-\zeta d/dz} f(z) = \gamma\left(-\frac{\tau}{2i\pi} \frac{d}{dz}\right) f(z), \end{aligned}$$

that is

$$b = \gamma\left(-\frac{\tau}{2i\pi} \frac{d}{dz}\right).$$

Thus, a direct operator translation of the tau-binomial theorem reads

$$(12) \quad \gamma(1 + \tau - z) \gamma\left(-\frac{\tau}{2i\pi} \frac{d}{dz}\right) \gamma(z) = \gamma\left(-\frac{\tau}{2i\pi} \frac{d}{dz}\right) \gamma(1 + \tau - z) \gamma(z) \gamma\left(-\frac{\tau}{2i\pi} \frac{d}{dz}\right),$$

but before you say a word, here is another, not so direct but shorter one.

Apply the reflection formula (5) to a different part of (7') to obtain

$$(7''') \quad \hat{\gamma}(z - y) \gamma(y) = \gamma(z) \int_{\text{NW}}^{\text{SE}} d\zeta e^{i\pi(\zeta - z)(1 + \tau - \zeta - z)/\tau} \hat{\gamma}(z - \zeta) \hat{\gamma}(\zeta - y),$$

or

$$bc = ceb,$$

where operators  $b$  and  $c$  are the same as above, and  $e$  acts as

$$ef(z) = \int_{\text{NW}}^{\text{SE}} d\zeta e^{i\pi(\zeta - z)(1 + \tau - \zeta - z)/\tau} \hat{\gamma}(z - \zeta) f(\zeta).$$

It is then easy to figure out that

$$e = \gamma\left(-\frac{\tau}{2i\pi} \frac{d}{dz} + z - \frac{1 + \tau}{2}\right)$$

and thus obtain Kashaev's 'pentagon identity'<sup>4</sup>

$$(13) \quad \gamma\left(-\frac{\tau}{2i\pi} \frac{d}{dz}\right) \gamma(z) = \gamma(z) \gamma\left(-\frac{\tau}{2i\pi} \frac{d}{dz} + z - \frac{1 + \tau}{2}\right) \gamma\left(-\frac{\tau}{2i\pi} \frac{d}{dz}\right).$$

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<sup>4</sup>No, it is not called that because it features five factors. See (Kashaev 2000) for an explanation and further references.

If this is still not good enough, rewrite (7) as

$$(7''''') \quad \gamma(\lambda) \psi_\lambda(z) = \gamma(z) \int_{\text{NW}}^{\text{SE}} d\zeta \, \hat{\gamma}(z - \zeta) \psi_\lambda(\zeta),$$

or

$$\gamma(\lambda) \psi_\lambda = cb \psi_\lambda,$$

where  $b$  and  $c$  are the same as above, and  $\psi_\lambda(z) = e^{-2i\pi\lambda z/\tau} \gamma(z - \lambda)$  are the same eigenfunctions that have already appeared twice on these pages (Sections 1 and 4). So, this is just another spectral equation on functions that already satisfy two. Hence, by comparing the respective eigenvalues here and, say, in the first of equations (3), we have

$$\gamma\left(-\frac{\log(e^{-2i\pi z} + e^{\tau d/dz})}{2i\pi}\right) = \gamma(z) \gamma\left(-\frac{\tau}{2i\pi} \frac{d}{dz}\right),$$

or, more compactly,

$$(14) \quad X(e^{-2i\pi z} + e^{\tau d/dz}) = X(e^{-2i\pi z}) X(e^{\tau d/dz}),$$

where  $X$  is the function such that

$$X(e^{-2i\pi z}) = \gamma(z),$$

that is

$$\begin{aligned} X(w) &= \gamma\left(-\frac{\log w}{2i\pi}\right) = \frac{(q^2 w; q^2)_\infty}{(w^{1/\tau}; q^{-2/\tau^2})_\infty} \\ &= \frac{(1 - q^2 w)(1 - q^4 w)(1 - q^6 w) \dots}{(1 - w^{1/\tau})(1 - q^{-2/\tau^2} w^{1/\tau})(1 - q^{-4/\tau^2} w^{1/\tau}) \dots}. \end{aligned}$$

This is called ‘Schützenberger’s equation’ after the famous French combinatorialist who discovered such noncommutative exponentiality fifty years ago. He did without those scary  $1/\tau$ -th powers though. I will explain after a remark.

Remember we were wondering how the same function could emulate the Gamma and exponential functions at the same time? Now we know. The  $X$ -function may have zeros, poles and a cut, but it is the exponential property that counts, and on this grounds alone it should be accepted as a legitimate ‘noncommutative’ exponential function. It must be stressed, though, that the exponential property itself has also become ‘noncommutative’. What happens, then, if the same factors are multiplied the other way around? As it turns out, this:

$$(15) \quad X(e^{\tau d/dz}) X(e^{-2i\pi z}) = X(e^{-2i\pi z} - e^{-2i\pi z} e^{\tau d/dz} + e^{\tau d/dz}).$$

The derivation is quite similar to that of Schützenberger’s equation (14) and is therefore left as an exercise.

## 7. Breakdown

An explanation is indeed in order and not just of where Schützenberger fits in all this, but, more broadly, of how that all-important exponential property (14) could be derived step by step rather than, as above, pulled out of the hat. Let us, then, rederive it in that sort of heuristic  $q$ -algebraic style typical of the subject.

Recall (or rederive) that operators  $e^{-2i\pi z}$  and  $e^{\tau d/dz}$  satisfy Weyl’s relation

$$e^{-2i\pi z} e^{\tau d/dz} = q^2 e^{\tau d/dz} e^{-2i\pi z}$$

and, for now, forget all else. That is, consider instead formal operators  $u$  and  $v$  only subject to the relation

$$(i) \quad uv = q^2 vu.$$

Apply the latter repeatedly to show that

$$\begin{aligned}
& (u - uv + v)(1 - q^2v)(1 - q^4v)(1 - q^6v) \dots \\
&= (1 - q^2v)(u - q^2uv + v)(1 - q^4v)(1 - q^6v) \dots \\
&= (1 - q^2v)(1 - q^4v)(u - q^4uv + v)(1 - q^6v) \dots \\
&= \dots = (1 - q^2v)(1 - q^4v)(1 - q^6v) \dots (u + v),
\end{aligned}$$

that is  $(u - uv + v)(q^2v; q^2)_\infty = (q^2v; q^2)_\infty(u + v)$ , and similarly that  $(u + v) \times (q^2u; q^2)_\infty = (q^2u; q^2)_\infty(u - uv + v)$ , and therefore

$$\begin{aligned}
(u + v)(q^2u; q^2)_\infty(q^2v; q^2)_\infty &= (q^2u; q^2)_\infty(u - uv + v)(q^2v; q^2)_\infty \\
&= (q^2u; q^2)_\infty(q^2v; q^2)_\infty(u + v).
\end{aligned}$$

Thus,  $(q^2u; q^2)_\infty(q^2v; q^2)_\infty$  commutes with  $u + v$ , and must therefore be its function. Call it  $F$  and set  $u = 0$  and  $v = w$  – which is permitted by relation (i) of course – to see that

$$F(w) = F(0 + w) = (0; q^2)_\infty(q^2w; q^2)_\infty = (q^2w; q^2)_\infty.$$

Hence what actually was Schützenberger’s discovery:<sup>5</sup>

$$(q^2(u + v); q^2)_\infty = (q^2u; q^2)_\infty(q^2v; q^2)_\infty,$$

or in words, the numerators alone already satisfy Schützenberger’s equation.

Turning to the denominators, we seem to be stuck, because formal operators may only be raised to positive integer powers – which  $1/\tau$  is not. Still, note that for positive integers we have

$$u^m v^n = q^{2mn} v^n u^m,$$

and assume, for lack of a better idea, that this somehow remains true if one or both powers are no longer integer. Then, whatever  $u^{1/\tau}$  and  $v^{1/\tau}$  might really be, they are bound, on one hand, to satisfy Weyl’s relation with  $q^{-1/\tau^2}$  instead of  $q$ ,

$$(ii) \quad v^{1/\tau} u^{1/\tau} = q^{-2/\tau^2} u^{1/\tau} v^{1/\tau},$$

and on the other, to commute with  $u$  and  $v$ :

$$(iii) \quad uv^{1/\tau} = q^{2/\tau} v^{1/\tau} u = e^{2i\pi\tau/\tau} v^{1/\tau} u = v^{1/\tau} u \quad \text{and} \quad u^{1/\tau} v = \dots = vu^{1/\tau}.$$

But what about  $(u + v)^{1/\tau}$  then? Note that

$$(uv^{-1})(u + v) = q^2(u + v)(uv^{-1}),$$

and therefore, by the same little trick that gave us relations (iii), we have

$$(uv^{-1})(u + v)^{1/\tau} = q^{2/\tau}(u + v)^{1/\tau}(uv^{-1}) = (u + v)^{1/\tau}(uv^{-1}).$$

Thus,  $(u + v)^{1/\tau}$  commutes with something that is not (a series in)  $u + v$ , and therefore with both  $u$  and  $v$  separately. Then it is a series in  $u^{1/\tau}$  and  $v^{1/\tau}$ , but the only such series to scale right is  $u^{1/\tau} + v^{1/\tau}$ . Hence

$$(iv) \quad (u + v)^{1/\tau} = u^{1/\tau} + v^{1/\tau},$$

and the rest is straightforward. By relation (ii) and the same argument as for the numerators we have

$$(v^{1/\tau} + u^{1/\tau}; q^{-2/\tau^2})_\infty = (v^{1/\tau}; q^{-2/\tau^2})_\infty(u^{1/\tau}; q^{-2/\tau^2})_\infty,$$

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<sup>5</sup>By the way, the crucial group-likeness property of Drinfel’d’s universal  $sl_2$  R-matrix,

$$\Delta \otimes \text{id}(R) = R^{13} R^{23} \quad \text{and} \quad \text{id} \otimes \Delta(R) = R^{13} R^{12},$$

is really just this formula in fancy disguise. See (Faddeev 2000) and (Bytsko Teschner 2003) for more on the Quantum Group connection.



then relation (iv) turns this into

$$((u+v)^{1/\tau}; q^{-2/\tau^2})_\infty = (v^{1/\tau}; q^{-2/\tau^2})_\infty (u^{1/\tau}; q^{-2/\tau^2})_\infty,$$

and in their turn relations (iii) allow to reunite the numerators with denominators and obtain

$$X(u+v) = X(u)X(v),$$

as we want. It only remains, therefore, to find out if those hypothetical relations (ii-iv) actually hold good if formal  $u$  and  $v$  are replaced back by

$$u = e^{-2i\pi z} \quad \text{and} \quad v = e^{\tau d/dz}.$$

But obviously<sup>6</sup>

$$u^{1/\tau} = e^{-2i\pi z/\tau} \quad \text{and} \quad v^{1/\tau} = e^{d/dz},$$

and therefore relation (ii) holds as good as (i), relations (iii) are checked trivially, and, finally, (iv) has already been established back in Section 1 (relation (4)). So we are done – but another important point now needs clearing up.

As we have just learnt, the numerators alone already satisfy Schützenberger’s equation – and with it in fact all the other noncommutative identities in question and a full complement of so called  $q$ -hypergeometric sums very similar to our integral identities, only much older.<sup>7</sup> So, what we have actually shown so far is that adding suitable denominators does no harm. But what good does it do? The answer is already apparent in Figure 1. Note that if  $|\tau| = 1$ ,<sup>8</sup> then, on top of central symmetry, zeros and poles of the  $\gamma$ -function are mirror symmetric to each other about the line passing through 1 and  $\tau$ , and as a result  $|\gamma(z)| = 1$  everywhere on that line – none of which can be said of the numerator because it has no poles in the first place. So, if we are to go beyond formal algebra and develop any kind of a unitary theory, then, as it was first realized by L. Faddeev, we really need the whole of the  $\gamma$ -function, and  $|\tau| = 1$  is the case to look into.<sup>9</sup>

## 8. Case $|\tau| = 1$

For convenience, let us adjust the ‘reference frame’ so that the aforementioned symmetry axis becomes the real line. To this end, fix “Planck’s constant”  $\hbar$  and offset  $\omega''$ , set  $\omega = \sqrt{-\pi\hbar/2\tau}$  and  $\omega' = \sqrt{-\pi\hbar\tau/2}$  – so that conversely

$$\hbar = -\frac{2\omega\omega'}{\pi} \quad \text{and} \quad \tau = \frac{\omega'}{\omega},$$

and redefine the  $\gamma$ -function like this:

$$\gamma_{\text{new}}(z) = \gamma_{\text{old}}\left(\frac{z + \omega''}{2\omega}\right).$$

In these terms, the original setup corresponds to  $\omega'' = 0$  and  $\hbar = -\tau/2\pi$ , but now we opt instead for  $\omega'' = \omega + \omega'$  and some positive  $\hbar$ , say,  $\hbar = 1/2\pi$  for a change. The infinite product expansion and defining equations then read

$$(16) \quad \gamma(z) = \frac{(1 + e^{-i\pi(z-\omega')/\omega})(1 + e^{-i\pi(z-3\omega')/\omega})(1 + e^{-i\pi(z-5\omega')/\omega}) \dots}{(1 + e^{-i\pi(z+\omega)/\omega'})(1 + e^{-i\pi(z+3\omega)/\omega'})(1 + e^{-i\pi(z+5\omega)/\omega'}) \dots},$$

$$(17) \quad \frac{\gamma(z + \omega')}{\gamma(z - \omega')} = 1 + e^{-i\pi z/\omega} \quad \text{and} \quad \frac{\gamma(z + \omega)}{\gamma(z - \omega)} = 1 + e^{-i\pi z/\omega'},$$

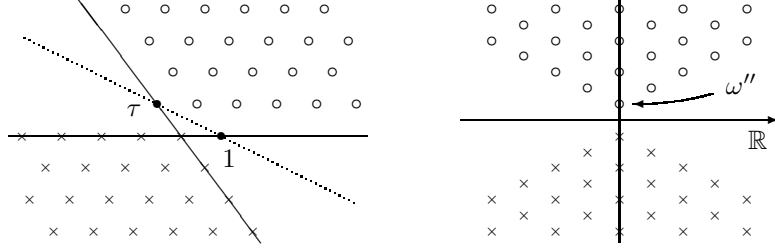
zeros/poles are located at the points  $z = k\omega + l\omega'$  with  $k$  and  $l$  positive/negative odd integers, and  $\gamma(z) \sim 1$  as  $z \rightarrow \infty$  in the sector  $|\arg z| < \arg \omega = (\pi - \arg \tau)/2$

<sup>6</sup>See footnote to Lemma 2 below.

<sup>7</sup>See (Koornwinder 1996) for details and history.

<sup>8</sup>Not to be mistaken for  $|q| = 1$ .

<sup>9</sup>In fact, the limit case when  $\tau > 0$  would do as well, but we have to choose something.

FIGURE 2. Case  $|\tau| = 1$  in old and new frames.

or in particular as  $z \rightarrow +\infty$ . In their turn, the reflection formula (5), tau-gamma integral (6) and tau-binomial theorem (7) become

$$(18) \quad \gamma(z)\gamma(-z) = \alpha e^{i\pi z^2},$$

$$(19) \quad \gamma(z + \omega'') = \beta \int_{-\infty}^{\infty} d\zeta \frac{e^{-2i\pi\zeta z}}{\gamma(\omega'' - \zeta)},$$

$$(20) \quad \frac{\gamma(y)\gamma(z + \omega'')}{\gamma(y + z)} = \beta \int_{-\infty}^{\infty} d\zeta \frac{e^{-2i\pi\zeta z} \gamma(y - \zeta)}{\gamma(\omega'' - \zeta)},$$

where in both integral the real line is suitably indented,<sup>10</sup> and, if you must know,  $\alpha = -\tau e^{-i\pi\omega''^2} \gamma(\omega - \omega')^2$  and  $\beta = \gamma(\omega - \omega')/2\omega$ , and in its turn,  $\gamma(\omega - \omega')$  is what used to be  $\gamma(1)$  (see Sections 2 and 4). And, of course, on top of all this we have the proto-unitarity property

$$|\gamma(z)| = 1 \quad \text{for all } z \in \mathbb{R},$$

which implies that operator  $\gamma(A)$  is unitary whenever  $A$  is self-adjoint, and allows to restate our findings so far as follows. Identities (12) and (13) become

**Theorem 1.** *Let  $q$  and  $p$  be (the self-adjoint closures in  $L^2(\mathbb{R})$  of) Schrödinger's position and momentum operators*

$$qf(z) = zf(z) \quad \text{and} \quad pf(z) = \frac{f'(z)}{2i\pi}$$

*or a unitary equivalent pair. Then operators  $\gamma(\pm q)$ ,  $\gamma(p)$  and  $\gamma(p + q)$  are all unitary and satisfy the '3 = 4 identity'*

$$(21) \quad \gamma(-q)\gamma(p)\gamma(q) = \gamma(p)\gamma(-q)\gamma(q)\gamma(p)$$

*and 'pentagon identity'*

$$(22) \quad \gamma(p)\gamma(q) = \gamma(q)\gamma(p + q)\gamma(p).$$

Relation (4) becomes

**Lemma 2.** *Let  $u(t)$  and  $v(t)$  be Weyl–Stone–von–Neumann's operators*

$$u(t) = e^{2i\pi tq} \quad \text{and} \quad v(t) = e^{2i\pi tp},$$

*and let*

$$u \equiv u(2\omega') = e^{-i\pi q/\omega} \quad \text{and} \quad v \equiv v(2\omega') = e^{-i\pi p/\omega}.$$

*Then*

$$u^{1/\tau} = u(2\omega) = e^{-i\pi q/\omega'} \quad \text{and} \quad v^{1/\tau} = v(2\omega) = e^{-i\pi p/\omega'},$$

*and furthermore*

$$(23) \quad (u + v)^{1/\tau} = u^{1/\tau} + v^{1/\tau},$$

<sup>10</sup>See footnote to (7).

provided the branch is so chosen that  $(e^{-i\pi z/\omega})^{1/\tau} = e^{-i\pi z/\omega'}$  for all  $z \in \mathbb{R}$ .<sup>11</sup>

Finally, identities (14) and (15) become

**Theorem 3.** *If function  $X$  is such that  $X(e^{-i\pi z/\omega}) = \gamma(z)$  for all  $z \in \mathbb{R}$ , that is*

$$X(w) = \frac{(1+qw)(1+q^3w)(1+q^5w)\dots}{(1+q^{-1/\tau^2}w^{1/\tau})(1+q^{-3/\tau^2}w^{1/\tau})(1+q^{-5/\tau^2}w^{1/\tau})\dots}$$

*with the same proviso about the branch, then operators  $X(u)$ ,  $X(v)$ ,  $X(u+v)$  and  $X(u+qvu+v)$  are all unitary and satisfy ‘Schützenberger’s identity’*

$$(24) \quad X(u+v) = X(u)X(v)$$

*and the other way around identity*

$$(25) \quad X(v)X(u) = X(u+qvu+v).$$

It is straightforward to upgrade the arguments of Section 6 to the level of strict proofs.<sup>12</sup> It should be noted though that it was mostly for demonstration purposes that in that Section every identity was independently derived all the way from the tau-binomial theorem. It would be more practical to derive only one of them and then transform it into the remaining three by purely ‘noncommutative’ techniques. For instance, the pentagon identity (22) can be easily transformed into  $3 = 4$  by either of the following ways.

Recall

**Lemma 4** (folklore). *Operators*

$$\sigma_1 = \alpha e^{iq^2/2\hbar} \quad \text{and} \quad \sigma_2 = \alpha e^{ip^2/2\hbar}$$

*satisfy Artin’s braid group relation*<sup>13</sup>

$$(26) \quad \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2.$$

*Proof.* By the product differentiation rule we have

$$p e^{iq^2/2\hbar} = e^{iq^2/2\hbar} (p+q),$$

and by unitary equivalence

$$(p+q) e^{ip^2/2\hbar} = e^{ip^2/2\hbar} q.$$

Hence

$$p e^{iq^2/2\hbar} e^{ip^2/2\hbar} = e^{iq^2/2\hbar} e^{ip^2/2\hbar} q,$$

and the result follows at once.  $\square$

Now, ‘divide’ Artin’s relation by the pentagon identity (22),<sup>14</sup>

$$\frac{1}{\gamma(q)} \frac{1}{\gamma(p)} \sigma_1\sigma_2\sigma_1 = \frac{1}{\gamma(p)} \frac{1}{\gamma(p+q)} \frac{1}{\gamma(q)} \sigma_2\sigma_1\sigma_2,$$

then use the product differentiation rule (see the above proof) and reflection formula (18) to simplify the left hand side like this:

$$\begin{aligned} \alpha^3 \frac{1}{\gamma(q)} \frac{1}{\gamma(p)} e^{iq^2/2\hbar} e^{ip^2/2\hbar} e^{iq^2/2\hbar} \\ = \alpha^3 \frac{e^{iq^2/2\hbar}}{\gamma(q)} e^{ip^2/2\hbar} \frac{e^{iq^2/2\hbar}}{\gamma(q)} = \gamma(-q)\gamma(p)\gamma(-p)\gamma(-q), \end{aligned}$$

<sup>11</sup>Fittingly, such a branch only fails to exist if  $\omega$  is real – which it is absolutely not.

<sup>12</sup>See also (Woronowicz 2000) and (Bytsko Teschner 2003) for alternative takes on the subject.

<sup>13</sup>By the way, these two triple products equal not only each other in fact, but also  $(\sqrt{i}$  times) Fourier transform understood as a unitary operator in  $L^2(\mathbb{R})$ , that is  $\sqrt{i} e^{i\pi(q^2+p^2-1/2\pi)}$ . I leave it to the reader to figure this out.

<sup>14</sup>I write  $A/B$  for  $AB^{-1}$  whenever  $A$  and  $B$  commute.

and the right hand side like this:

$$\begin{aligned} \alpha^3 \frac{1}{\gamma(p)} \frac{1}{\gamma(p+q)} \frac{1}{\gamma(q)} e^{ip^2/2\hbar} e^{iq^2/2\hbar} e^{ip^2/2\hbar} \\ = \alpha^3 \frac{e^{ip^2/2\hbar}}{\gamma(p)} \frac{e^{iq^2/2\hbar}}{\gamma(q)} \frac{e^{ip^2/2\hbar}}{\gamma(-p)} = \gamma(-p)\gamma(-q)\gamma(p). \end{aligned}$$

Hence  $\gamma(-p)\gamma(-q)\gamma(p) = \gamma(-q)\gamma(p)\gamma(-p)\gamma(-q)$ , which is the  $3 = 4$  identity (12) modulo unitary equivalence. So, one way to transform the pentagon identity (22) into  $3 = 4$ , or vice versa for that matter, is use the formula

$$(3 = 4 \text{ identity}) = (\text{pentagon identity})^{-1}(\text{Artin's relation}).$$

The other transformation is, in contrast, one way only, and it goes like this:

$$\begin{aligned} \gamma(-q)\gamma(p)\gamma(q) &= \gamma(-q)\gamma(q)\gamma(p+q)\gamma(p) = \gamma(q)\gamma(-q)\gamma(p+q)\gamma(p) \\ &= \gamma(q)\gamma(p+q)\gamma(p)\gamma(-q) = \gamma(p)\gamma(-q)\gamma(q)\gamma(p). \end{aligned}$$

This time all is done with the pentagon relation, which is first used ‘as is’, then in its unitary equivalent form

$$\gamma(-q)\gamma(p+q) = \gamma(p+q)\gamma(p)\gamma(-q),$$

and then again ‘as is’.

## 9. Yang-Baxterization

The bottom line so far is that we have obtained every operator interpretation of the tau-binomial theorem (7) I know of. It only remains, then, to do the same to the ultimate integral identity (11). Unfortunately, due to the greater number of variables involved, there are more such interpretations than would be appropriate in an introductory article. We will, therefore, leave Theorem 3 for another time and limit ourselves to generalization of Theorem 1 and Lemma 4. Here it is.

**Theorem 5.** *For all  $\lambda, \mu \in \mathbb{R}$  there hold quasi-Yang-Baxter equations*

$$\begin{aligned} (27) \quad \gamma(p)\gamma(\lambda-p)\gamma(\mu+p-q)\gamma(\lambda-p+q)\gamma(q)\gamma(\mu-q) \\ = \gamma(\mu-q)\gamma(q)\gamma(p+q)\gamma(\lambda+\mu-p-q)\gamma(p)\gamma(\lambda-p) \end{aligned}$$

$$(28) \quad \frac{\gamma(-q)}{\gamma(\lambda-q)} \frac{\gamma(p)}{\gamma(\lambda+\mu+p)} \frac{\gamma(q)}{\gamma(\mu+q)} = \frac{\gamma(p)}{\gamma(\mu+p)} \frac{\gamma(-q)\gamma(q)}{\gamma(\lambda-q)\gamma(\mu+q)} \frac{\gamma(p)}{\gamma(\lambda+p)}$$

and the true Yang-Baxter equation

$$(29) \quad \sigma_1(\lambda)\sigma_2(\lambda+\mu)\sigma_1(\mu) = \sigma_2(\mu)\sigma_1(\lambda+\mu)\sigma_2(\lambda),$$

where  $\sigma(\lambda)$  is Fateev-Zamolodchikov’s *R-matrix*.<sup>15</sup>

$$\sigma_1(\lambda) = \frac{\sigma_1}{\gamma(\frac{\lambda}{2}+q)\gamma(\frac{\lambda}{2}-q)} \quad \text{and} \quad \sigma_2(\lambda) = \frac{\sigma_2}{\gamma(\frac{\lambda}{2}+p)\gamma(\frac{\lambda}{2}-p)}.$$

*Proof.* With some patience, all three identities could be derived starting from the ultimate integral identity (11) and following the guidelines of Section 6, which is left as another exercise. This would not quite prove the theorem though, for, as we remember, the said integral identity has not been actually verified. We need, therefore, some kind of a direct ‘noncommutative’ proof, and this is where the techniques shown in the previous section come into their own.

<sup>15</sup>It is called that after its finite-dimensional relative from (Fateev Zamolodchikov 1982).

Apply the pentagon identity to the underlined pieces either as is or in a suitable unitary equivalent form:

$$\begin{aligned}
& \gamma(p) \gamma(\lambda - p) \gamma(\mu + p - q) \gamma(\lambda - p + q) \gamma(q) \gamma(\mu - q) \\
&= \gamma(p) \gamma(\mu + p - q) \gamma(\lambda + \mu - q) \gamma(\lambda - p) \gamma(\lambda - p + q) \gamma(q) \gamma(\mu - q) \\
&= \gamma(p) \gamma(\mu + p - q) \gamma(\lambda + \mu - q) \gamma(q) \gamma(\lambda - p) \gamma(\mu - q) \\
&= \gamma(p) \gamma(\mu + p - q) \gamma(\mu - q) \gamma(q) \gamma(\lambda + \mu - q) \gamma(\lambda + \mu - p - q) \gamma(\lambda - p) \\
&= \gamma(\mu - q) \gamma(p) \gamma(q) \gamma(\lambda + \mu - q) \gamma(\lambda + \mu - p - q) \gamma(\lambda - p) \\
&= \gamma(\mu - q) \gamma(q) \gamma(p + q) \gamma(p) \gamma(\lambda + \mu - q) \gamma(\lambda + \mu - p - q) \gamma(\lambda - p) \\
&= \gamma(\mu - q) \gamma(q) \gamma(p + q) \gamma(\lambda + \mu - p - q) \gamma(p) \gamma(\lambda - p).
\end{aligned}$$

This settles (27), and then (28) follows in exactly the same way as in the last section the  $3 = 4$  followed from the pentagon identity, that is by ‘dividing’ Artin’s relation (26) by (27). In its turn, the Yang-Baxter equation emerges if (26) is divided, instead of (27), by its unitary equivalent variant

$$\begin{aligned}
& \gamma(\frac{\lambda}{2} + p) \gamma(\frac{\lambda}{2} - p) \gamma(\frac{\lambda + \mu}{2} + p - q) \gamma(\frac{\lambda + \mu}{2} - p + q) \gamma(\frac{\mu}{2} + q) \gamma(\frac{\mu}{2} - q) \\
&= \gamma(\frac{\mu}{2} - q) \gamma(\frac{\mu}{2} + q) \gamma(\frac{\lambda + \mu}{2} + p + q) \gamma(\frac{\lambda + \mu}{2} - p - q) \gamma(\frac{\lambda}{2} + p) \gamma(\frac{\lambda}{2} - p).
\end{aligned}$$

And finally, if you have done the exercise suggested at the beginning of the proof, you can reverse it and thus settle (11).  $\square$

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